

DEVIATION FROM CLASSICAL ELASTICITY IN THE ACOUSTIC RESPONSE OF AUXETIC FOAMS

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ABSTRACT

Acoustic properties of an auxetic foam were measured to validate a previously proposed lower bound for Poisson's ratio, $\nu \geq 1/5$, inferred from classical two-parameter elasticity theory. This limit differs from the commonly reported lower limit of -1 for isotropic materials in the linear elastic range. For a foam sample measured to have a value of $\nu > 0.2$, agreement was found between the measured flexural resonance frequency of a disk sample and the theoretical value. On the other hand, for an auxetic sample ($\nu < 0$), the prediction from the two-parameter theory was significantly in error. Thus, for materials having $\nu < 1/5$ (auxetic foams, as well as very hard solids such as diamond, germanium and fused quartz), the equations of classical elasticity are invalid.

INTRODUCTION

In order to derive structure-property relationships of general validity, most laboratory characterizations of materials are carried out in the linear regime; that is, the sample is in a state of equilibrium and the applied perturbation does not drive the material far from equilibrium. For dynamic measurements, "not far" means the system is confined to states attainable by spontaneous thermal fluctuations. For elastic mechanical measurements, linearity implies proportionality of stress and strain. The requirement for a linear, reversible mechanical response determines the amplitude limit for the perturbation. When linearity prevails and there is no relaxation (no dependence on the rate of loading), the usual assumption is that the equations of classical elasticity^{1,2} can be employed. These constitutive equations relate the various elastic constants, obtained by experiment, to one another. For an isotropic material, only two of these constants are unique, although a large number can be defined in order to describe different experimental configurations. Classical elasticity theory enables any constant to be calculated from any other two. This paper addresses the lower bound for ν in order for the classical theory to be applicable.

Poisson's ratio is the only elastic constant that is not a ratio of stress to strain or its inverse. Table 1 shows the relation of ν to the other common, elastic constants.³ These equations are obtained from classical elasticity. The lower bound on all moduli in Table 1 is non-zero finite in order to comply with the

second law of thermodynamics (non-negative strain energy); however, the lower bound on ν is less certain. Expressing ν in terms of the bulk modulus B and the shear modulus G (describing respective changes in volume and shape)

$$B = \frac{2(1+\nu)}{3(1-2\nu)}G \quad (1)$$

the obtained range is¹

$$-1 < \nu < 1/2 \quad (2)$$

Previously we addressed the upper bound on ν .⁴ For real materials, $\nu < 0.5$, so that the assumption of incompressibility for rubber is only an approximation. Methods of handling “near compressibility” in modeling and finite element analysis have been discussed.^{5,6} Of interest herein is the predicted lower bound of -1.

Since all elastic constants are equally valid, pairs other than B and G can be used to derive limits on ν . The most restrictive limit must be the correct one, since relations involving all other constants would be satisfied. The classical elasticity equation relating the longitudinal, M , and biaxial, H , moduli is

$$\nu = \frac{M}{2H+4M} \left[2\frac{H}{M} - 1 \pm \left(9 - 8\frac{H}{M} \right)^{1/2} \right] \quad (3)$$

This quadratic equation has two roots; the one consistent with the behavior of most solids (>99%) yields⁷

$$1/5 \leq \nu < 1/2 \quad (4)$$

The inference is that for any material having a Poisson’s ratio less than the lower bound of 0.2 in eq.(4), classical elasticity is violated and thus inapplicable.^{3,7} Ironically, those few materials known to have $\nu < 0.2$, such as diamond, beryllium, and fused quartz, are very hard, which makes mechanical characterization difficult. As a result, it is for these materials that recourse is made to the classical elasticity expressions to calculate the desired engineering constants. That is, the theory is commonly applied in situations for which it appears to be invalid.

In this work we describe acoustic measurements on open-cell foams. Foams can have a complicated stress-strain response, especially for deformations involving both tension and compression, since the latter may induce buckling of the cell walls.⁸ By compressing thermoplastic foams above their softening point and releasing the pressure after cooling, foams can be prepared in which the cell walls are pre-buckled.⁹ Subsequent stretching causes de-buckling that results in lateral expansion of the specimen, causing $\nu < 0$.¹⁰ In a previous study^{11,12} we showed that for foams with $\nu \geq 0.2$, the measured ν was in good agreement with the value calculated using the classical elasticity equation

$$E = 2G(\nu + 1) \quad (5)$$

However, for auxetic foams ($\nu < 0$) it was found that eq. (5) underestimates ν by as much as 36%. Thus, this earlier work^{11,12} corroborated the restricted limits on ν given by eq. (4), rather than the usually stated range (eq. (2)).

Given the general acceptance of the classical theory, it is useful to further validate the newly proposed lower bound of 1/5. Toward that end, we carried out acoustic measurements on two foams, having either ν within the range of eq. (4) or below the lower bound of 0.2. Our expectation that calculated engineering properties will be in error for the latter, in accord with the inapplicability of the classical theory for materials having $\nu < 0.2$, is borne out by the acoustic experiments.

EXPERIMENTAL

The polyurethane foam (McMaster-Carr) was thermoplastic with a glass transition temperature around -50°C . It had an open cell structure with an average pore size of *ca.* 0.4 mm (Figure 2). The material was processed to give auxetic behavior using the following procedure: isotropic compression for 2 hours at 110°C (5 deg. above the melting point of the hard domains), followed by cooling to room temperature while under compression. The dimensional recovery was no more than 10%, and the structure was stable at RT. The density of the foam (0.048 g/ml initially) increases with degree of compression, as shown in Figure 1. Compressions equal to or greater than two were determined to yield auxetic foam. Optical micrographs of the auxetic foam used for acoustic measurements is also shown in Figure 2. Note that the slight orientation of the elliptical pores in the as-received foam are absent after compression, although the structure of the compressed foam is difficult to discern, as observed in other open-cell polyurethane auxetic foams.¹³

The material after processing was mechanically anisotropic, but transversely isotropic. Young's modulus and Poisson's ratio were measured on samples elongated on an Instron 5500R at a strain rate = 0.002 s^{-1} , the low rate causing relaxation to be negligible. This stretching was carried out parallel to the compression direction, as were the stress-strain measurements. Displacements were determined from fiducial marks, with a minimum of three sets of marks in both the longitudinal and transverse directions used to calculate the two elastic constants.

Acoustic measurements on the polyurethane samples were done in an air-filled Bruel & Kjaer acoustic impedance tube. Circular disks were cut to fit firmly inside of the impedance tube (29 mm inner diameter), with a 4-microphone configuration¹⁴ used to measure the reflection and transmission characteristics over the frequency range of 0.5 – 6 kHz.

RESULTS

From the initial slope of the stress-strain curves we obtain the Young's moduli listed in Table 2. Poisson's ratios were determined by measuring the lateral displacement as a function of extension (Figure 3). There is a small increase with increasing strain; however, the magnitude of the change is not greater than the uncertainty in the data. Thus, by extrapolating to zero strain we obtain the values given in Table 2, with the error bars corresponding to the assumption of invariance of ν with strain.

To obtain the elastic behavior of the foams from acoustic testing, flexural resonances were identified in the spectral data obtained with the acoustic impedance tube. The flexural resonance frequency corresponds to a narrow band of enhanced transmission through the sample, as observed, for example, using a similar setup with silica aerogel.¹⁵ Using this approach, the flexural resonance frequencies can be related to the elastic properties of the foam based on the classical elasticity equations (Mindlin theory¹⁶ for the vibration of a thick elastic disc); that is,¹⁵

$$f_{res} = \frac{1}{4\pi\sqrt{3}} \frac{c_p h}{a^2} \frac{1}{R+S} \left[\sqrt{1+2\pi^2(R+S)} - 1 \right] \quad (6)$$

where h is the thickness, a the radius, and the coefficients R and S defined by

$$R = \frac{1}{12} \left(\frac{h}{a} \right)^2, \quad S = \frac{1}{6\pi(1-\nu)\kappa^2} \left(\frac{h}{a} \right)^2 \quad (7)$$

Here, $\kappa = \pi/\sqrt{12}$ is the shear correction factor, and $c_p = \left(E / \rho(1-\nu^2) \right)^{1/2}$ is the compressional plate wave speed.

Table 2 compares the calculated and measured f_{res} . There is agreement within the experimental uncertainty for the foam having $\nu = 0.2$. For the auxetic foam, f_{res} calculated using eq. (6) is a factor of two smaller than the measured value. This is consistent with the inapplicability of eq. (6), derived from classical elasticity, for materials having $\nu < 0.2$; i.e., outside the range in eq. (4).

SUMMARY

The acoustic results presented herein corroborate the failure of classical elasticity whenever $\nu < 1/5$, as previously demonstrated from measurements of the Young's and shear moduli of auxetic foams¹¹. Validation of this lower bound requires measurements of three elastic constants. Usually, eq. (2) is assumed to be correct, and only two constants are experimentally determined. For example, in earlier work various vibrational mode frequencies were used to deduce the shear or bulk moduli, with the latter used as fitting parameters.¹⁷ But since only the fundamental frequency depends on G or B , but not the

change in resonant frequency with mode number, deviations from classical elastic are not evident; rather, the failure of the classical theory only manifests as undetected error in the values deduced for the elastic constants.

The sample studied herein had $\nu = -0.51$, which is much less than 0.2; thus, deviations from the classical theory were substantial. This contrasts with a prior study of fused quartz for which $\nu = 0.17$.¹⁸ No discrepancies between measured and calculated elastic constants were apparent, presumably because the deviations from classical elasticity were too small.¹⁸ Potential errors in the present experiments, due to nonlinearity, viscoelasticity, or an inhomogeneous foam structure, have been minimized. To wit, (i) the elastic constants were extrapolated to zero strain, with any uncertainty assessed by comparison to the value obtained by averaging over all strains; (ii) the strain rate employed was sufficiently slow to maintain mechanical equilibrium, as judged from the reversibility of the stress-strain curve; and (iii) fiducial mark displacements were averaged over about 40,000 cells to avoid problems due to foam inhomogeneity. Of course, any deviations from the classical constitutive laws due to the design of the experiments would affect equally the analysis of the conventional ($\nu = 0.2$) foam; however, the calculated f_{res} for this sample was in accord with the experimental values.

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Table 1. Isotropic elastic relations involving Poisson's ratio.

(ν, G, E)	$\frac{E}{G} = 2(1 + \nu)$	(ν, B, M)	$\frac{B}{M} = \frac{1 + \nu}{3(1 - \nu)}$	(I, ν, G)	$\frac{G}{I} = \frac{1}{2}(1 - 2\nu)$
(ν, G, B)	$\frac{G}{B} = \frac{3(1 - 2\nu)}{2(1 + \nu)}$	(H, ν, G)	$\frac{H}{G} = \frac{2(1 + \nu)}{1 - \nu}$	(I, ν, E)	$\frac{E}{I} = (1 - 2\nu)(1 + \nu)$
(ν, G, M)	$\frac{G}{M} = \frac{1 - 2\nu}{2(1 - \nu)}$	(H, ν, E)	$\frac{E}{H} = 1 - \nu$	(I, ν, B)	$\frac{B}{I} = \frac{1}{3}(1 + \nu)$
(ν, E, B)	$\frac{E}{B} = 3(1 - 2\nu)$	(H, ν, B)	$\frac{H}{B} = \frac{3(1 - 2\nu)}{1 - \nu}$	(I, ν, M)	$\frac{M}{I} = 1 - \nu$
(ν, E, M)	$\frac{E}{M} = \frac{(1 - 2\nu)(1 + \nu)}{1 - \nu}$	(H, ν, M)	$\frac{H}{M} = \frac{(1 - 2\nu)(1 + \nu)}{(1 - \nu)^2}$	(I, H, ν)	$\frac{H}{I} = \frac{(1 - 2\nu)(1 + \nu)}{1 - \nu}$

E \equiv Young's modulus; G \equiv shear modulus; B \equiv bulk modulus; M \equiv longitudinal modulus;
 H \equiv biaxial stress modulus; I \equiv biaxial strain modulus

Table 2. Mechanical and acoustic results for foams.

compression ratio	ν	ρ (g/ml)	E (kPa)	f_{res} (kHz)		error
				measured	eq. (6)	
1	0.20 ± 0.01	0.048	232 ± 8	1.34	1.38	3%
3.0	-0.51 ± 0.11	0.128	76 ± 19	1.01	0.55	84%

FIGURES

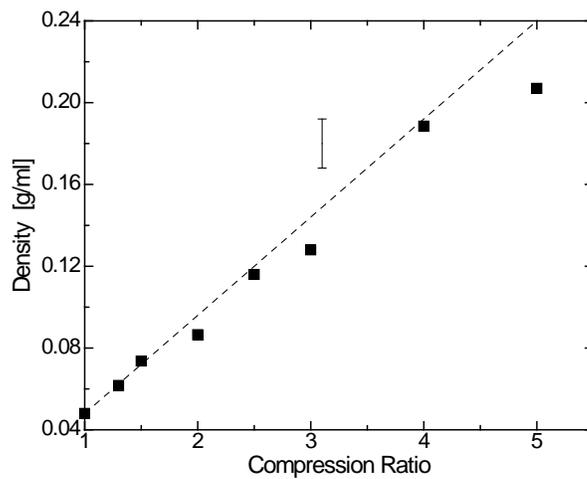


Figure 1. Density of foam as a function of the volume compression during high temperature annealing. The error bar indicates sample variability. There is modest recovery, reflected in values below the dashed line. For compressions of two or higher, $\nu < 0$.

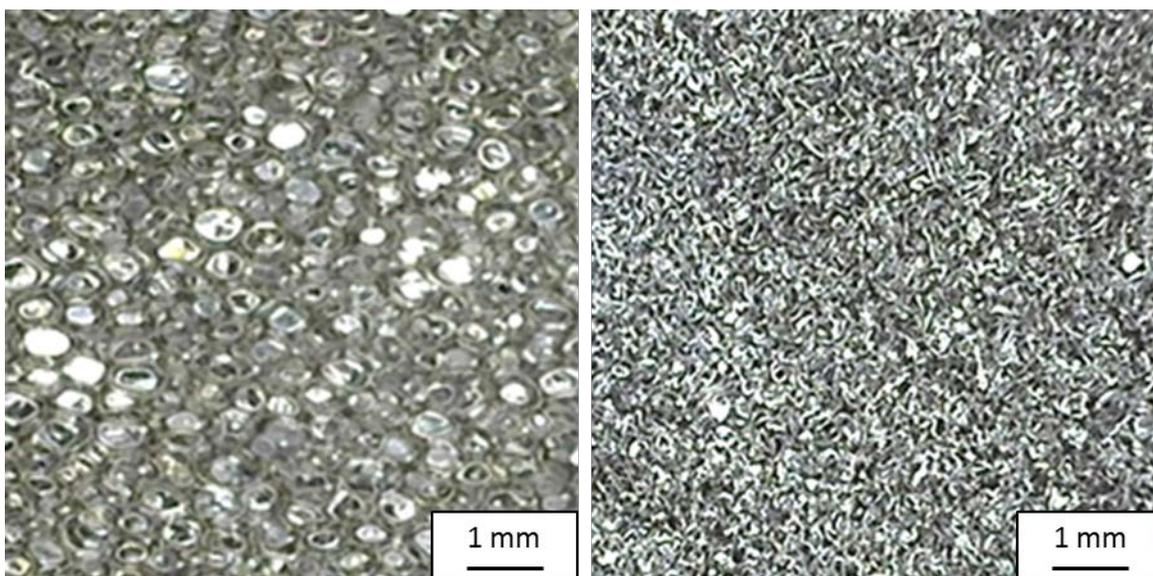


Figure 2. Optical micrographs showing (left) the as-received foam and (right) the auxetic material. Orientation of the cells are evident on the left, while the morphology of the auxetic foam was essentially isotropic.

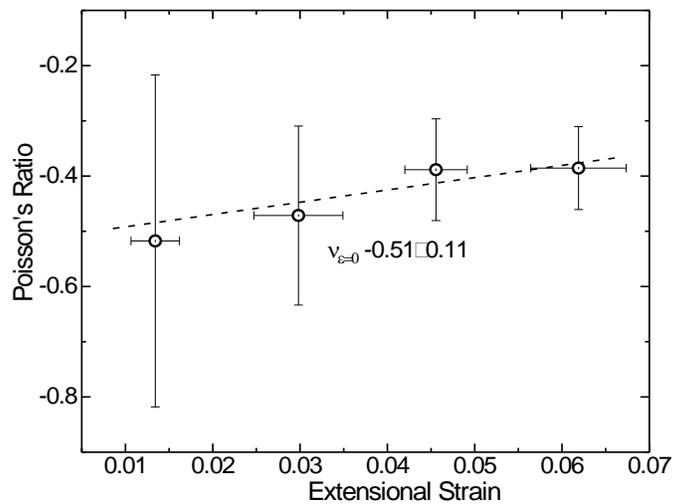


Figure 3. Poisson's ratio measured for the compressed foam as a function of extension. Extrapolation to zero strain gives -0.51 , which differs by 0.11 from the average value assuming no influence of strain on Poisson's ratio. The error bars are the limit of error.