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Limits to Poisson's ratio in isotropic materials—general result for arbitrary deformation

P H Mott and C M Roland

Chemistry Division, Naval Research Laboratory, Code 6120, Washington, DC 20375-5342, USA

E-mail: peter.mott@nrl.navy.mil and mike.roland@nrl.navy.mil

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Abstract

The lower bound customarily cited for Poisson's ratio ν , -1 , is derived from the relationship between ν and the bulk and shear moduli in the classical theory of linear elasticity. However, experimental verification of the theory has been limited to materials having $\nu \geq 0.2$. From consideration of the longitudinal and biaxial moduli, we recently determined that the lower bound on ν for isotropic materials from this theory is actually $\frac{1}{5}$. Herein we generalize this result, first by analyzing expressions for ν in terms of six common elastic constants, and then by considering arbitrary strains. The results corroborate that $\nu \geq \frac{1}{5}$ for classical linear elasticity to be applicable. Of course, a few materials exist for which $\nu < 0.2$, thus deviating from this bound; accurate analysis of their mechanical behavior requires more sophisticated elasticity models.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The ratio of lateral strain ε_{22} to longitudinal strain ε_{11} defines the elastic constant

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} \quad (1)$$

for a material under uniaxial stress σ_{11} . This constant is named for Poisson, who defined it in 1829 in his single constant theory of linear elasticity, in which $\nu = \frac{1}{4}$ for all solids [1]. Recent interest in auxetic materials ($\nu < 0$) [2, 3] and nano-composites, in which Poisson's ratio is used to characterize mechanical behavior [4–8], has renewed attention to this quantity.

Much of the experimental investigations of the mechanical behavior of isotropic solids in the early 19th century were devoted to measuring ν , in order to verify the single constant Poisson idea. Its refutation developed sporadically; the first evidence appeared in 1848, when ν was found to be ca. $\frac{1}{3}$ for various oxide glasses and brasses [9], and in 1859, when experiments determined $\nu = 0.295$ for steel [10]. Unfortunately, other less accurate measurements supported the theory, and the controversy persisted into

the 1860s. Lamé's two-constant linear elasticity model for isotropic materials [11] was adopted by most researchers soon thereafter, in part because it accommodates variation in ν [12–14]; however, this did not prove the theory was correct.

According to the classical theory, for an isotropic material only two elastic constants are unique, so its validation requires measurement and comparison of three different constants. For example, the relation

$$\nu = \frac{1}{2} - \frac{E}{6B} \quad (2)$$

can be used to compare measured values of Poisson's ratio to that determined from Young's modulus E and the bulk modulus B . This approach presents two challenges: (i) highly precise data are required (see review [15]); and (ii) conventional solids are often nonlinear even at strains as small as 10^{-5} [16, 17]. Experimental verification appeared in the early 1900s [18, 19], with data for iron, tin, aluminum, copper, silver, platinum and lead [19] conforming to the two-constant theory (figure 1). In the past 100 years, the application of this generalized Hooke's law has been fully accepted and is universally applied in science and

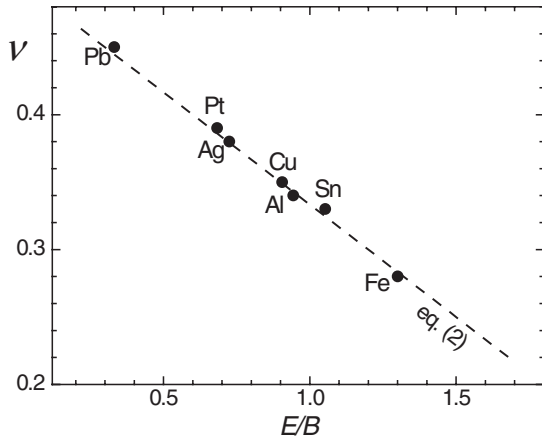


Figure 1. Experimental data from Grüneisen [19], demonstrating the validity of Lamé’s quadratic theory of linear elasticity for $\nu \geq 1/5$.

engineering. Thus, it is common practice to limit characterizations of isotropic solids to two elastic constants, obtained for example from shear and longitudinal wave speed measurements [20, 21], with other parameters calculated from the classical elasticity relations.

The accepted theoretical limits on Poisson’s ratio are much lower than the experimental range in figure 1, which means that the theory has actually been verified only for those materials having $\nu \geq 0.2$. The conventional limits are found from [12]

$$G = B \frac{3(1 - 2\nu)}{2(1 + \nu)}, \quad (3)$$

where G is the shear modulus. To minimize the strain energy at equilibrium and avoid spontaneous deformation, G and B must be positive, leading to the oft-stated ‘thermodynamically admissible’ range [12, 22]

$$-1 < \nu < \frac{1}{2}. \quad (4)$$

This derivation of the limits on ν is the obvious one, considering deformations involving changes in size and shape, but other elastic constants are equally valid. The actual thermodynamic limits on ν have never been determined experimentally, and measurements for isotropic materials occupy a much narrower range than the conventional limits (figure 1). Reviews of the literature of more than 3000 measurements on 596 different substances over a wide range of temperature and pressure, including pure elements, engineering alloys, polymers, ceramics and glasses, show that with very few exceptions (e.g. porous quartz or very hard materials such as diamond and beryllium), $\nu \geq 0.2$ for isotropic, homogeneous materials [23, 24]. Thus, the lower limit in equation (4) does not represent the behavior of most real materials. This does not mean that real materials cannot have $\nu < 0.2$, but only that the Lamé theory has not been experimentally validated for $\nu < 0.2$.

Notwithstanding its conceptual appeal, there is no mathematical or physical justification for preferring G and B over other pairs of constants to determine the limits on Poisson’s ratio. For example, using the classical relation for ν in terms of the biaxial modulus, H , and the longitudinal modulus, M (see table 1), we have shown from the roots of

a quadratic expression that the range in equation (4) is split into [24]

$$-1 < \nu \leq \frac{1}{5}, \quad (5a)$$

$$\frac{1}{5} \leq \nu < \frac{1}{2}. \quad (5b)$$

Since elastic properties are unique, only one range can be valid; moreover, the upper limit of $\frac{1}{2}$ agrees with experimental data. Thus, this more restrictive upper range, $\frac{1}{5} \leq \nu < \frac{1}{2}$, appears to be the correct limit for classical linear elasticity, since values of ν still conform to equation (4). The argument might be made that the range extending to $-1 < \nu$ in equation (5) is mathematically allowed, and hence represents an acceptable bound. However, rejection of spurious roots is common when an analysis produces two or more solutions; physical considerations are applied to eliminate roots that are false. Examples include the Landau–Lifshitz equation for the motion of a charge [25], analysis of projectile trajectories in air [26], Pythagoras’ theorem for right triangles, and more generally in the solutions of ordinary differential equations [27]. We also note that a recent theoretical analysis [28], based on symmetry arguments from elastic constants that were restricted to linear combinations of the two Lamé constants, similarly found expressions for ν having multiple roots; the lower bound on Poisson’s ratio was larger than -1 , namely $(1 - \sqrt{2})/2$. Thus, two recent analyses [24, 28] undermine the traditional range of ν for the generalized Hooke’s law to be valid.

This more restrictive lower bound on Poisson’s ratio in equation (5b) is important because it means that whenever a material has $\nu < 0.2$, the equations of linear elasticity cannot apply; a more sophisticated, and as yet undeveloped model of elasticity must be invoked to provide relations between elastic constants for that material. In this work we first extend the analysis of Mott and Roland [24] to all commonly defined elastic constants, in order to obtain their associated limits for Poisson’s ratio. We then generalize these results to any deformation. Our previous conclusion [24], that the minimum of ν for an isotropic material is $\frac{1}{5}$, is shown to be general for materials for which the traditional equations (e.g. Lamé elasticity model) are valid.

2. Limits on ν from common elastic constants

For an isotropic solid with axial strains ϵ_{11} , ϵ_{22} , ϵ_{33} and shear strains γ_{12} , γ_{23} , γ_{13} , the reversible work of deformation is [12]

$$2W = (\lambda + 2\mu) (\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 + \mu (\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2 - 4\epsilon_{22}\epsilon_{33} - 4\epsilon_{33}\epsilon_{11} - 4\epsilon_{11}\epsilon_{22}), \quad (6)$$

where λ and $\mu (= G)$ are the Lamé constants. (Note defining the quantities as tensorial shear strains, a factor of $\frac{1}{2}$ would be included for the shear strains on the right-hand side [14].) Differentiating equation (6) with respect to the strains defines the stress tensors

$$\begin{aligned} \frac{\partial W}{\partial \epsilon_{11}} &= \sigma_{11} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\epsilon_{11}, \\ \frac{\partial W}{\partial \gamma_{12}} &= \sigma_{12} = \mu\gamma_{12}, \quad \text{etc} \end{aligned} \quad (7)$$

Table 1. Relations between elastic constants that include Poisson's ratio.

Relation	Equation	Restrictions on ν
$\nu = \frac{3B - 2G}{6B + 2G}$	(T1)	$-1 < \nu < \frac{1}{2}$
$\nu = \frac{H - 3B}{H - 6B}$	(T2)	$-1 < \nu < \frac{1}{2}$
$\nu = \frac{3B - M}{3B + M}$	(T3)	$-1 < \nu < 1$
$\nu = \frac{H - 2G}{H + 2G}$	(T4)	$-1 < \nu < 1$
$\nu = \frac{M - 2G}{2M - 2G}$	(T5)	$-\infty < \nu < \frac{1}{2}$
$\nu = \frac{1}{2} - \frac{E}{6B}$	(T6)	$-\infty < \nu < \frac{1}{2}$
$\nu = \frac{1}{2} - \frac{G}{I}$	(T7)	$-\infty < \nu < \frac{1}{2}$
$\nu = 1 - \frac{E}{H}$	(T8)	$-\infty < \nu < 1$
$\nu = 1 - \frac{M}{I}$	(T9)	$-\infty < \nu < 1$
$\nu = \frac{E}{2G} - 1$	(T10)	$-1 < \nu < \infty$
$\nu = \frac{3B}{I} - 1$	(T11)	$-1 < \nu < \infty$
$\nu = \frac{1}{4} \left[\frac{E}{M} - 1 \pm \left(\frac{E^2}{M^2} - 10 \frac{E}{M} + 9 \right)^{\frac{1}{2}} \right]$	(T12)	$0 < \frac{E}{M} \leq 1: -1 < \nu < 0$ or $0 < \nu < \frac{1}{2}$
$\nu = \frac{1}{4} \left[\frac{H}{I} - 1 \pm \left(\frac{H^2}{I^2} - 10 \frac{H}{I} + 9 \right)^{\frac{1}{2}} \right]$	(T13)	$0 < \frac{H}{I} \leq 1: -1 < \nu < 0$ or $0 < \nu < \frac{1}{2}$
$\nu = -\frac{1}{4} \left[1 \mp \left(9 - 8 \frac{E}{I} \right)^{\frac{1}{2}} \right]$	(T14)	$0 < \frac{E}{I} \leq \frac{9}{8}: -1 < \nu \leq -\frac{1}{4}$ or $-\frac{1}{4} \leq \nu < \frac{1}{2}$
$\nu = \frac{M}{2H + 4M} \left[2 \frac{H}{M} - 1 \pm \left(9 - 8 \frac{H}{M} \right)^{\frac{1}{2}} \right]$	(T15)	$0 < \frac{H}{M} \leq \frac{9}{8}: -1 < \nu \leq \frac{1}{5}$ or $\frac{1}{5} \leq \nu < \frac{1}{2}$

When uniaxial loading is substituted (e.g. $\sigma_{11} \neq 0$ and all other $\sigma_{ij} = 0$) and defining Young's modulus as $E = \sigma_{11}/\varepsilon_{11}$, the following relations between the elastic constants are found:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad \nu = \frac{\lambda}{2\lambda + 2\mu}. \quad (8)$$

This algebraic operation can be carried out for any deformation or loading geometry, using the corresponding stiffness, to identify relations between elastic constants [29]. These relations are combined to obtain relations between the elastic constants, valid for all types of loading. For example, when substituting longitudinal loading (e.g. $\varepsilon_{11} \neq 0$ and all other strains = 0) and defining the longitudinal modulus as

$M = \sigma_{11}/\varepsilon_{11}$, we obtain

$$M = \frac{1 - \nu}{(1 - 2\nu)(1 + \nu)} E. \quad (9)$$

Table 1 lists all of the equations for Poisson's ratio from commonly defined moduli. Included are expressions that involve the biaxial stress modulus H , defined when $\sigma_{11} = \sigma_{22} \neq 0$ and all other $\sigma_{ij} = 0$, and the biaxial strain modulus I , defined when $\varepsilon_{11} = \varepsilon_{22} \neq 0$ and all other strains = 0. I is unusual, but is included here as the counterpart to H . The second column in the table 1 shows the restrictions on ν arising from the requirement that all elastic moduli are greater than zero. It is seen that the conventional limits, $-1 < \nu < \frac{1}{2}$, follow from equations (T1) and (T2). The other

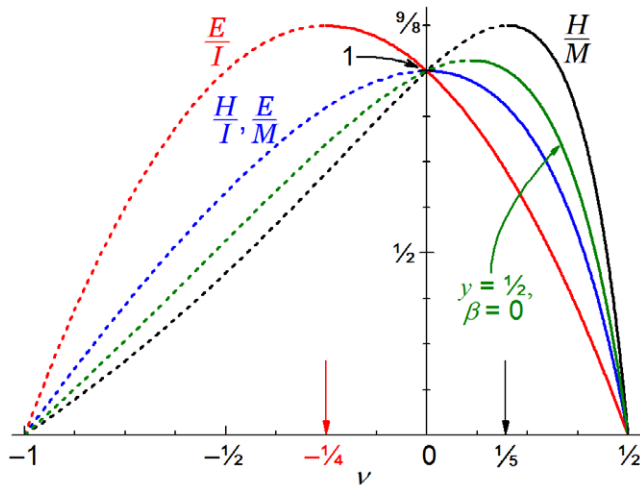


Figure 2. Poisson’s ratio as a function of the ratio of the indicated elastic constants, with positive roots shown indicated by the solid lines and negative roots with dashed lines. Also included are the two roots of equation (13) with $y = 1/2$ and $\beta = 0$. The limits encompassing all moduli are $1/5 \leq \nu < 1/2$.

linear expressions lead to wider ranges for ν . The most restrictive limits for Poisson’s ratio are the governing range, since all broader ranges are also satisfied (as is, of course, the requirement that the strain energy be non-negative).

Of special interest are the four quadratic relations, equations (T12)–(T15). These arise from stress–strain counterparts, such as E (defined from a stress) and M (defined from a strain). Note that if E/M is substituted for H/I , equation (T13) becomes equation (T12), and therefore the two equations are identical; thus,

$$\frac{E}{M} = \frac{H}{I}. \quad (10)$$

Each quadratic relation in table 1 has two roots that limit the span of Poisson’s ratio. These relations are plotted in figure 2, with the positive roots denoted by a solid line and the negative with a dashed line. For all expressions, the roots lie within the range -1 to $1/2$ and converge at smoothly continuous maxima. Restricting ν to real numbers means that

1. equations (T12) and (T13): $0 < E/M \leq 1$ with the same range for H/I . The two roots of this expression have ranges $-1 < \nu \leq 0$ and $0 \leq \nu < 1/2$. This equation also produces real values if $E/M \geq 9$, which has two roots with ranges $1 < \nu \leq 2$ and $2 \leq \nu < \infty$; however, this solution is discarded because it falls beyond the bounds of equation (4);
2. equation T14: $0 < E/I \leq 9/8$; the two roots have the ranges $-1 < \nu \leq -1/4$ and $-1/4 \leq \nu < 1/2$;
3. equation T15: $0 < H/M \leq 9/8$; the two roots have the ranges $-1 < \nu \leq 1/5$ or $1/5 \leq \nu < 1/2$.

There are companion relations for G and B , and these quadratic equations have interconnected roots. For example, the counterpart to equation (T15) for the bulk modulus is

$$B = \frac{M}{6} \left[3 \pm \left(9 - 8 \frac{H}{M} \right)^{1/2} \right] \quad (11)$$

and, having the same argument for the square root as in equation (T15), restricts $0 < H/M \leq 9/8$ for this expression to be real. The negative root has the range $0 < B/M \leq 1/2$, $1/2 \leq B/M < 1$ for the positive root. It can be shown that the positive root is linked to the positive root of equation (T15) and vice versa; that is, if $1/2 \leq B/M < 1$, then $1/5 \leq \nu < 1/2$.

Quadratic expressions with two possible solutions for G , B and ν are at odds with the behavior of real materials, which have unique elastic constants for any thermodynamic state. Therefore, only one set of solutions can be valid.

3. Limits on ν for arbitrary deformations

The considered elastic constants—shear G , hydrostatic pressure or dilatation B , uniaxial stress E , uniaxial strain M , biaxial stress H and biaxial strain I —permute a single stress or strain through the available loading combinations for an isotropic material. However, the possibility exists that more restrictive limits on ν can be found from other elastic constants derived from more complex combinations of stress or strain. To examine this, we introduce two, continuously variable elastic constants. The first is a biaxial stress with $\sigma_{11} \neq 0$ and $\sigma_{22} = y\sigma_{11}$, where y is a constant describing the fraction of biaxial stress, $0 \leq y \leq 1$; all other $\sigma_{ij} = 0$. The elastic constant for this variable stress geometry is

$$H_y = \frac{E}{1 - y\nu}. \quad (12)$$

When $y = 0$ (uniaxial loading), $H_0 = E$; when $y = 1$ (biaxial stress), equation (12) becomes equation (T8).

For the second constant, consider a variable biaxial strain $\varepsilon_{11} \neq 0$ and $\varepsilon_{22} = \beta\varepsilon_{11}$, where β is the fraction of biaxial strain, $0 \leq \beta \leq 1$; and all other strains = 0. The elastic constant for this variable strain geometry is

$$I_\beta = \frac{1 - \nu(1 - \beta)}{1 - \nu} M. \quad (13)$$

Similarly, when $\beta = 0$ (longitudinal deformation), $I_0 = M$, and when $\beta = 1$, equation (13) becomes equation (T9), corresponding to biaxial strain. These expressions define the elastic stiffness for any mixture of one- or two-dimensional stress or strain.

From the equations in table 1, many other relations that involve H_y and I_β can be derived. Of particular interest is

$$\nu = \frac{I_\beta}{4I_\beta + 2y(1 - \beta)H_y} \left\{ (1 - \beta + y) \frac{H_y}{I_\beta} - 1 \pm \left[9 - (10 - 2\beta - 2y + 4\beta y) \frac{H_y}{I_\beta} + (1 - \beta - y)^2 \frac{H_y^2}{I_\beta^2} \right]^{1/2} \right\}. \quad (14)$$

This equation combines the four quadratic expressions for Poisson’s ratio into a single, continuous function. Equation (14) does not imply that biaxial stress and biaxial strain conditions coexist; clearly they cannot. Rather, a material can be subjected to biaxial stress, and ν determined. In a separate experiment, the material can be subjected to biaxial strain, and ν determined. These two determinations of

ν have to be internally consistent, but if $\nu < 0.2$ is allowed, contradictions arise.

Each of the four quadratic expressions for ν , (T12)–(T15) in table 1, can be recovered by substituting the respective values for y and β into equation (14). Intermediate values y and β produce curves that lie between these extremes. Shown in figure 2 is the curve for $y = \frac{1}{2}$ and $\beta = 0$, which falls between the H/M and E/M curves. Likewise, the two roots of equation (14) meet without discontinuity. This common point is defined as $\nu^*(y, \beta)$ at H_y^*/I_β^* , which divides Poisson’s ratio into the ranges $-1 < \nu \leq \nu^*$ and $\nu^* \leq \nu < \frac{1}{2}$. Since the upper span corresponds to experimental data [30, 31], it is of interest to determine the lower limit ν^* . This point is found when the two roots are equal, which occurs when

$$9 - (10 - 2\beta - 2y + 4\beta y) \frac{H_y^*}{I_\beta^*} + (1 - \beta - y)^2 \left(\frac{H_y^*}{I_\beta^*} \right)^2 = 0. \quad (15)$$

This expression has the solutions

$$\frac{H_y^*}{I_\beta^*} = \frac{5 - y - \beta + 2\beta y \pm 2[(\beta^2 - \beta - 2)(y^2 - y - 2)]^{\frac{1}{2}}}{(1 - \beta - y)^2}. \quad (16)$$

The positive root is rejected because it returns $H_y^*/I_\beta^* \geq 9$, producing $\nu > 1$, which is beyond the bounds from equation (4). Note this corresponds to $E/M \geq 9$, which was also discarded in section 2 above.

The values of ν^* satisfying equation (16) for given y and β have the range $-\frac{1}{4} < \nu^* \leq \frac{1}{5}$, with $\nu^* = 0$ for $y = \beta$. In terms of the common elastic constants, (i) $\nu^* = 1$ at $\beta = 0$, $y = 0$, corresponding to $\nu^*(E, M)$; (ii) $\nu^* = \frac{1}{5}$ at $\beta = 0$, $y = 1$, corresponding to $\nu^*(H, M)$; (iii) $\nu^* = -\frac{1}{4}$ at $\beta = 1$, $y = 0$, corresponding to $\nu^*(E, I)$; and (iv) $\nu^* = 0$ at $\beta = 1$, $y = 1$, corresponding to $\nu^*(H, I)$. Thus, equation (15) merges the ranges of ν for specific conditions of stress and strain (figure 1) into a single continuous function describing arbitrary stress and strain. Fractional values of y and β in equation (13) determine ν^* for any combination of two-dimensional stress or strain. Again, the most restrictive range is the correct range, because it accommodates the other ranges, and the lower bound for Lamé’s theory to be applicable is $\frac{1}{5}$ for any stress and strain.

Note that equation (16) is undefined when $\beta + y = 1$. For this condition, the solution for H_y^*/I_β^* is found by substituting $a - y = \beta$ and taking the limit $a \rightarrow 1$ by twice applying L’Hôpital’s rule. The result is

$$\frac{H_y^*}{I_\beta^*} = 1 - \frac{(1 - 2y)^2}{4(y^2 - y - 2)}. \quad (17)$$

This demonstrates that there is no discontinuity when $\beta + y = 1$.

The companion quadratic relations for G and B are

$$G = \frac{I_\beta}{4 - 8\beta} \left\{ 3 + (1 - \beta + y) \frac{H_y}{I_\beta} \mp \left[9 - (10 - 2\beta - 2y + 4\beta y) \frac{H_y}{I_\beta} + (1 - \beta - y)^2 \frac{H_y^2}{I_\beta^2} \right]^{\frac{1}{2}} \right\}, \quad (18)$$

$$B = \frac{I_\beta}{6 + 6\beta} \left\{ 3 - (1 - \beta + y) \frac{H_y}{I_\beta} \pm \left[9 - (10 - 2\beta - 2y + 4\beta y) \frac{H_y}{I_\beta} + (1 - \beta - y)^2 \frac{H_y^2}{I_\beta^2} \right]^{\frac{1}{2}} \right\}. \quad (19)$$

The inverted \pm sign in equation (18) denotes that its negative root is linked to the positive roots of equation (14) and (19).

4. Exceptions

As stated in the introduction, isotropic materials exist for which $\nu < \frac{1}{5}$, although they are rare. Homogenous materials which show this behavior include pyrite [32], α -cristobalite [33], diamond [34–36], a TiNb₂₄Zr₄Sn_{7.9} (β -type titanium) alloy [37], boron nitride [38], α -beryllium [39] and certain silicate glasses [40]. In the former cases (pyrite, cristobalite, diamond), elastic properties have been determined from vibrational measurements of single crystals and aggregate isotropic behavior is inferred. For the titanium alloy, boron nitride, beryllium and SiO₂ glasses, elastic properties of the aggregate were determined by vibrational methods, in which two elastic constants are measured, with Poisson’s ratio in turn found from the expressions in table 1. It can be seen that while homogeneous solids having $\nu < \frac{1}{5}$ have been identified, for none have the Lamé relations been tested.

There are recent reports of auxetic behavior in crystalline materials that exhibit negative ν in certain directions [41, 42]. However, when the aggregate isotropic behavior is examined, these substances show the conventional behavior, $\nu \geq \frac{1}{5}$. There is also a class of open-cell foams that have negative Poisson’s ratio, due to debuckling of the cell walls [3]. These auxetic foams exhibit nonlinear mechanical properties [43], so that the application of linear elasticity is problematic. Fitting their behavior to more complicated elasticity models has had limited success [44], although recently we showed that the equations of Lamé elasticity theory fail for such materials [45]. Recent investigations of larger scale, two-dimensional skeletal structures, both experimental [46, 47] and theoretical [48], also discovered auxetic behavior, but linear elasticity does not apply to deformations larger than mathematically infinitesimal, so that the theory cannot be tested.

5. Summary

The equations of classical elasticity (Lamé’s model) impose restrictions on the values of Poisson’s ratio, derived from the requirement that the strain energy be non-negative. Any pair of elastic constants leads to various expressions for the bounds on ν , but for mutual consistency, the most restrictive limits are the correct ones. The result, $\frac{1}{5} \leq \nu < \frac{1}{2}$, is shown to be the valid range for an isotropic material subjected to arbitrary loading or deformation. This range comports with the values of ν for the vast majority of isotropic materials, even though substances having $\nu < \frac{1}{5}$ do exist. However, the classical equations cannot be applied for the latter.

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