Failure of classical elasticity in auxetic foams

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Poisson's ratio, ν, was measured for four materials, a rubbery polymer, a conventional soft foam, and two auxetic foams. We find that for the first two materials, having $\nu \geq 0.2$, the experimental determinations of Poisson's ratio are in good agreement with values calculated from the shear and tensile moduli using the equations of classical elasticity. However, for the two auxetic materials ($\nu < 0$), the equations of classical elasticity give values significantly different from the measured ν. We offer an interpretation of these results based on a recently published analysis of the bounds on Poisson's ratio for classical elasticity to be applicable. *Copyright 2013 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution 3.0 Unported License.* [http://dx.doi.org/10.1063/1.4802925]

I. INTRODUCTION

Poisson's ratio (ν) is a constant that describes the transverse strain, ε_{22} or ε_{33}, of an elastic body accompanying a longitudinal strain, ε_{11}

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \tag{1}$$

For a mechanically isotropic material, Poisson's ratio is unique, having but one value.[1–3] The classical theory of elasticity for infinitesimal linear strain (i.e., Lamé's theory)[4] links ν of an isotropic solid to the other elastic constants, including the moduli and Lamé constants.[5,6] Because of the appeal of representing strain as the sum of a volumetric and a deviatoric (shear) strain, the most common expression involves the bulk, B, and shear, G, moduli[7]

$$G = B\frac{3(1 - 2\nu)}{2(1 + \nu)} \tag{2}$$

Eq. (2) and related expressions are commonly used to calculate an elastic constant of interest from elastic constants that are more amenable to direct experimental determination.

In this paper we carry out measurements on four materials, a rubbery polymer, a soft, conventional foam, and two foams exhibiting isotropic auxeticity, originating in de-buckling of cell ribs that leads to large changes in transverse dimensions during longitudinal elongation.[11,8] We compare Poisson's ratios measured directly on these materials, ν_{exp}, to values calculated from the classical expression

$$\nu_{calc} = \frac{E}{2G} - 1 \tag{3}$$

where E is Young's modulus. This work represents the first attempted corroboration of the equations for ν for values spanning a range nearly encompassing the classical bounds, $-1 < \nu < 0.5$. We find that for two materials herein for which $\nu \geq 0.2$, the equations of classical elasticity are accurate. However, for the two materials having $\nu < 0.2$, measurements of the dimensional changes during elongation yield values of ν that are significantly smaller than those calculated from eq. (3). This failure of classical elasticity is consistent with a recent prediction[5] that Lamé's theory is valid only when $\nu \geq 0.2$.

II. EXPERIMENTAL

The polyurethane (McMaster-Carr) was an open-cell foam with a density equal to 0.048 g/cm^3. The cell dimensions were measured to be 0.3 – 0.4 mm, with rib thickness ∼0.07 mm for the foam as received. Auxetic foams were prepared by triaxial compression of rectangular samples (initially either 46.4 mm × 46.4 mm × 217.5 mm or 51.2 mm × 51.2 mm × 240 mm) in a mold (dimensions = 32 mm × 32 mm × 150 mm), followed by heating above the softening point of the hard segments (∼105°C). Upon removal from the mold following slow cooling to room temperature, the foams showed ∼10 % expansion relative to the size of mold. The final linear compression ratios were 1.27 and 1.45. To ensure no adhesion of the cell walls, the samples were stretched 20% in each of three orthogonal directions. These two auxetic foams have small cell dimensions, *ca.* 0.2 – 0.3 mm. We designate the as-received foam PU1, and the auxetic samples as PU2 and PU3, where the number designates their respective volumetric compressive strains of 1, 2, and 3. A solid elastomer sample was also prepared by curing 1,4-polyisoprene (Natsyn from Goodyear) with 2% by weight dicumyl peroxide for 30 minutes at 160°C.

The tensile modulus and Poisson's ratio were measured on samples elongated on an Instron 5500R at a strain rate (= 0.002 s^{-1}) sufficiently slow to yield equilibrium values. Strains were determined from the displacement of fiducial marks on test specimens (initially 170 mm long and 36 mm wide), obtained from digitized photographs (Olympus E-PM1, 4032 × 3024 pixels), with the resolution defined by the pixel size of the fiducial images (∼12 μm). The software (Digplot; polymerphysics.net/software.html) used to analyse the images provided about 1/10 pixel resolution. At least three sets of two marks each per longitudinal and transverse direction were used to calculate ν and the Young's modulus

$$E = \frac{\sigma_e}{\varepsilon_e} \tag{4}$$

where σ_e and ε_e are the respective engineering stress and strain. Strains were measured over a 30 mm × 30 mm area, so that any inhomogeneities in the foams due to the cell structure were averaged out.

The shear modulus was measured with a sandwich configuration, also using the Instron at a shear strain rate equal to 0.002 s^{-1}; test samples were 50 mm long × 4 mm wide × 4 mm thick. Fiducial marks were again used to determine that strain, avoiding any errors due to end effects. To verify the shear measurements, G was also determined using a torsion geometry on ring specimens (25.4 mm outer diameter and 11.7 mm inner diameter) with an ARES rheometer operating at the same low shear rate. The shear modulus is given by

$$G = \frac{f t}{l w \delta} \tag{5}$$

where δ and f are the displacement, and force, respectively, and l, w, and t are the respective sample length, width, and thickness. The shear strain $\gamma = \delta / t$. For both uniaxial and shear measurements, the strain rates were confirmed to be sufficiently slower than the rate of relaxation, so that the samples were in mechanical equilibrium. Measurements at slower rates gave equivalent stresses.

III. RESULTS

The PU1 is transversely isotropic, having a modulus 60% higher in the thicker dimension; thus, we measured displacement of fiducial marks lying in the symmetric plane. The auxetic foams, PU2 and PU3, behave isotropically along all axes up to at least 5% strain. The deformation mechanism of the auxetic foams involves de-buckling of the cell ribs, which causes their modulus to be lower than that of the precursor material.[11,8]

Figure 1 shows Poisson's ratio measured directly for the polyisoprene and the three foams. The uncertainty in the data arises in part from our ability to resolve the fiducial images. The polyisoprene has a homogeneous structure, and typically elastomers have Poisson's ratio within the range 0.49 and 0.5.[9] We find no systematic variation for the polyisoprene over our range of strain measurements, obtaining $\nu_{exp} = 0.496$. The foams all show ν_{exp} that increases over the range of strains

TABLE I. Elastic constants and Poisson's ratio.

	polyisoprene	PU1	PU2[a]	PU3[a]
$E(\varepsilon = 0)$	1.145 ± 0.002	0.1450 ± 0.0003	0.0451 ± 0.0002	0.0502 ± 0.0002
$G(\gamma = 0)$	0.382 ± 0.003	0.0601 ± 0.0005	0.0550 ± 0.0002	0.0640 ± 0.0007
$G(\gamma = 0.005)$	—	—	0.0524 ± 0.0011	0.0605 ± 0.0010
$\nu_{calc} \; \gamma = 0$	0.499 ± 0.014	0.206 ± 0.012	-0.590 ± 0.020	-0.608 ± 0.023
ν_{exp}	0.496 ± 0.006	0.204 ± 0.006	-0.699 ± 0.008	-0.650 ± 0.015

[a]Stated error reflects uncertainty in fitted line used to extrapolate measurements to zero strain.
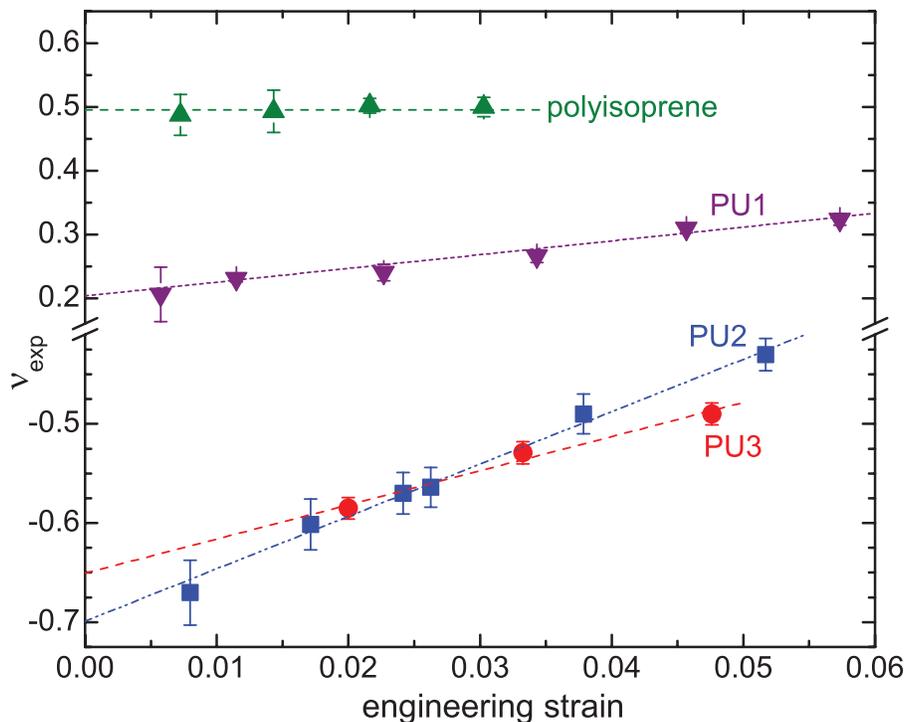

FIG. 1. Directly measured Poisson's ratio for the four materials. The data for the foams have some dependence on strain, with the value obtained by linear extrapolation to $\varepsilon = 0$ listed in Table I.

(*ca.* $1 - 5\%$ elongation). For the PU1 measured in the isotropic plane, ν_{exp} increases about 10% with strain, and linear extrapolation to zero strain gives $\nu_{exp} = 0.20$. Over this same range of strains the auxetic foams show an increase in ν_{exp} of about 20%. We extrapolate to zero strain by a linear fit to the data, obtaining $\nu_{exp} = -0.70$ and -0.65 for PU2 and PU3, respectively. The more compressed foam has a smaller (absolute value) of Poisson's ratio; similar results were reported previously for polyurethane foams.[8] The values of Poisson ratios determined by direct measurement of longitudinal and transverse deformations, ν_{exp}, are tabulated in Table I, along with the uncertainties.

Figure 2 displays Young's moduli for the foams; it was constant up to a few percent tensile strain for all materials. Also shown are the shear moduli, which showed some dependence on strain. Regression yields the zero strain values given in Table I, which are consistent with the shear moduli measured by torsional rheometry. Note that PU3, which has the greatest volume compression, has a larger shear modulus than PU2 (PU1 having intermediate values). Since the mechanical response involves de-buckling of the foam, there is no certainty that the measured behaviour extrapolates smoothly to zero strain. Thus, the limit of error on G for the foams (Table I) is taken as the difference between the value determined by extrapolation to zero strain and the value measured for the lowest strain; the actual errors are likely smaller.
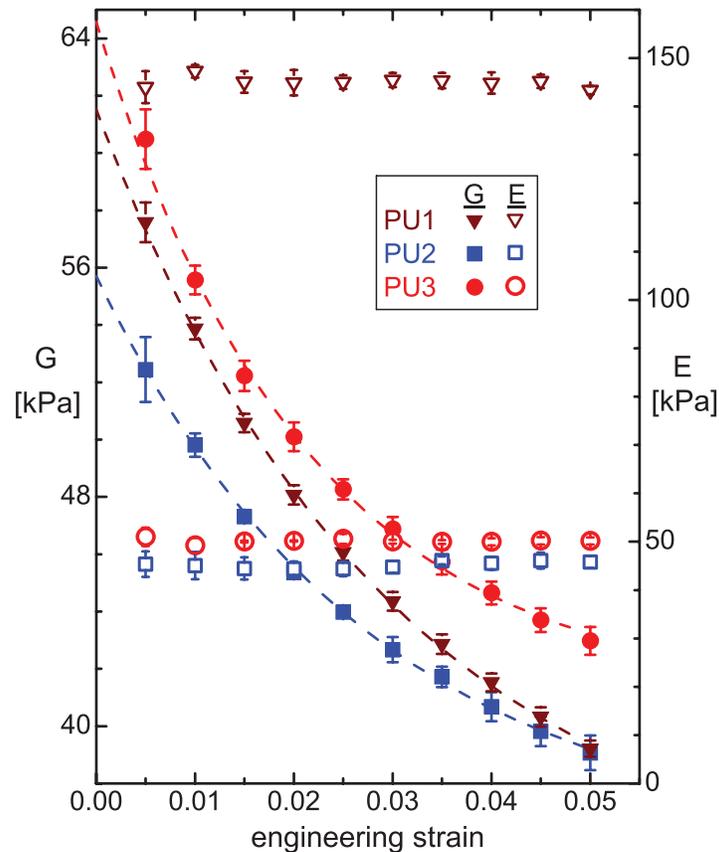
FIG. 2. Engineering modulus for the three foams as a function of strain. Young's moduli are constant; fitting the shear data to a first order polynomial gives the extrapolation to zero strain listed in Table I.

Table I shows Poisson's ratio calculated from the equation of classical elasticity (eq. (3)), using the measured values of the shear and Young's moduli. For polyisoprene and PU1, for which $\nu_{exp} \geq 0.2$, the difference between the calculated and experimental values is less than 1%; that is, the agreement is within the experimental uncertainties. This is expected and affirms the validity of our experimental methods. (Note that the transverse isotropy of PU1 did not result in deviations from classical elasticity.)

The situation is different for the two auxetic foams. For both PU2 and PU3, ν_{calc} underestimates the absolute value of the measured Poisson's ratio, by an amount (about 10%) that exceeds the experimental uncertainties (which are less than 3%). This conclusion that the calculated ν are larger than ν_{exp} does not depend on the method of extrapolating the data to low strain, as can be seen in Figure 3.

IV. DISCUSSION

Eq. (2), together with the requirement that the moduli are finite and positive, yields the well-known "classical" bounds on Poisson's ratio for isotropic materials[2]

$$-1 < \nu < 1/2 \tag{6}$$

It is known that anisotropic materials, e.g., foams having a honeycomb or otherwise novel structure,[10, 11] can exhibit ν that deviate from these limits, but the behaviour of such materials deviates from linear elasticity, so that eq. (6) does not apply. However, as we have pointed out recently,[5] with few exceptions ν for isotropic materials does not fall below 0.2.
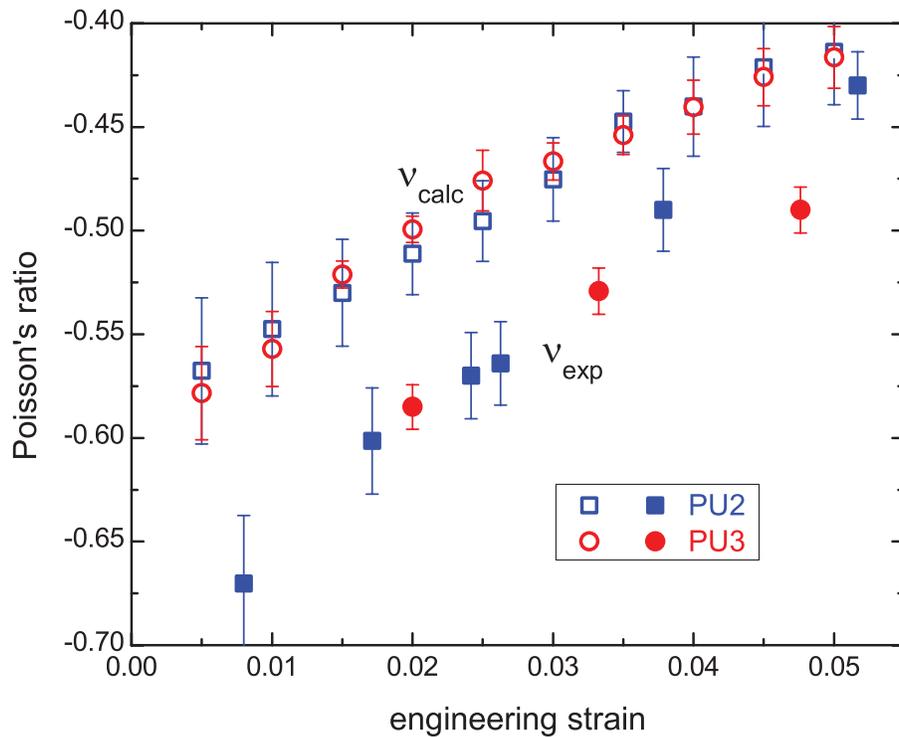
FIG. 3. Poisson's ratio for the auxectic foams measured (filled symbols) and calculated using eq. (3) (open symbols) as a function of the strain. Extrapolation to zero strain will not yield convergence.

For those materials having $\nu < 0.2$, our recent analysis[5,6] indicated that classical elasticity is inapplicable. Eq. (6) is arrived at by consideration of the bulk and shear moduli, but the intuitive appeal of expressing strains in terms of the bulk and deviatoric response does not elevate the significance of B and G above that of other elastic constants. Expressions for ν in terms of Young's modulus, the longitudinal modulus, the biaxial modulus, etc., lead to a larger lower limit on ν than eq. (5).[5,6] The argument in Ref. 5 is that the most restrictive bounds are the correct ones, since they do not yield discrepancies with any less restrictive limits. Accordingly, when classical elasticity applies, it was concluded that the limits on ν for classical elasticity are[5,6]

$$1/5 \leq \nu < 1/2 \tag{7}$$

This analysis accounts for the apparent failure of classical elasticity herein, although the present experimental results are valid regardless of the soundness of the derivation in Refs. 5 and 6.

Note that eq. (7) does not represent the limits for real materials, although the left-hand-side of eq. (7) has the virtue of corresponding to data for most substances. Eq. (7) was derived mathematically, using only the assumptions of Lamé's theory. The implication that the equations of classical elasticity cannot be used for materials having Poisson's ratio smaller than 0.2 is not a trivial result, because materials with $\nu < 0.2$ tend to be very hard (e.g., diamond,[12] beryllium,[13] and fused quartz).[14] This makes direct measurements difficult, with resort often made to techniques such as the resonant frequency method,[13,15,16] and the determined elastic constants are then used to calculate others.

V. SUMMARY

In this work we measured Poisson's ratio directly for four materials. For a homogeneous elastomer and a transversely isotropic foam, both having $\nu \geq 0.2$, Poisson's ratio calculated from the shear and tensile moduli is in agreement with the measured ν. However, for the two auxetic foams

the equation of classical elasticity overestimates ν, with the discrepancy exceeding the experimental error. Classical elasticity applies to small deformations for which the mechanical response is linear (e.g., strain energy quadratic in the strain), the behaviour is elastic,[2,8] and the material is sufficiently homogeneous for continuum mechanics to be applicable. Herein nonlinearity was only apparent for the shear modulus, and this was addressed in two ways: The data were extrapolated to zero strain or the value of G at the lowest measured strain was used; the difference between the two results yields a measure of the uncertainty. Moreover, it is not obvious how the ν_{exp} and ν_{calc} in Fig. 3 might converge in the low strain limit. The viscoelastic nature of the materials was rendered moot by using strain rates sufficiently slow that a state of mechanical equilibrium prevailed during the measurements. And material inhomogeneities arising from the cell structure of the foams were one hundred fold smaller than the fiducial distances; thus, the strain measurements were averaged *ca.* 5×10^5 cells.

For the auxetic foams Poisson's ratio is within the conventional limits of classical elasticity (eq. (6)), but beyond the more restrictive range of eq. (7). The discrepancy revealed herein with classical theory pertains to $\nu < 0.2$, which is the situation for which direct measurements are difficult, so that recourse is often made to the classical elasticity equations. Our results are consistent with a mathematical derivation[5,6] suggesting that eq. (7) does indeed give the proper classical bounds. The implication is that the use of the equations of classical elasticity to calculate ν or other elastic constants will be in error for any material for which Poisson's ratio is smaller than 0.2.

ACKNOWLEDGMENTS

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